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Note on invariants for uncoupled Ermakov systems

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Abstract

Several invariants for an Ermakov system consisting of a parametric oscillator and a Milne oscillator in one dimension are derived. This system appears in amplitude-phase analysis of parametric oscillator solutions. First- and second-order invariants for such a system are derived using the Wronskian relations of a fundamental pair of parametric oscillator solutions. In this way Wronskian invariants of the parametric oscillator system alone imply Ermakov–Lewis type invariants for the combined Ermakov system, as well as invariants for the Milne oscillator alone.

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Ermakov systems originated back in 1880 when Ermakov [1] studied a linear second-order ordinary differential equation (ODE) together with a particular nonlinear second-order ODE. The same pair of equations appeared as Milne [2] developed his amplitude-phase method for the time-independent Schrödinger equation. Several more general Ermakov systems have recently been developed (see [3–6] and cited references). In this note we consider time as the independent variable and the uncoupled Ermakov system takes the form

$$\ddot{x} + \omega^2(t)x = 0, \tag{1}$$

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}. \tag{2}$$

Equation (1) corresponds to a parametric oscillator with an arbitrary complex valued, time-dependent parameter $\omega^2(t)$ and (2), the Milne equation, describes the amplitude of a parametric oscillator solution derived from an amplitude-phase ansatz (suppressing the time dependence)

$$x = \rho \exp(\pm \mathrm{i}\phi), \tag{3}$$

with the phase satisfying

$$\dot{\phi} = \rho^{-2}. \tag{4}$$

The ansatz (3) clearly provides a link between the pair of equations (1) and (2), which explains how a particular solution of Milne's equation (2) provides a pair of fundamental solutions of (1). Furthermore, a general solution of (2) can be expressed in terms of fundamental solutions of (1) [7]. In later work Lewis and co-workers [8] found the canonical invariant (the Ermakov–Lewis invariant) for the system (1), (2)

$$\mathcal{L} = \frac{1}{2}\left((x\dot{\rho} - \dot{x}\rho)^2 + \left(\frac{x}{\rho}\right)^2\right), \tag{5}$$

which was also applied in quantum mechanical problems [9–11].

The main objective of the present analysis is to derive classical invariants of the Ermakov system from Wronskian invariants of solutions corresponding to equation (1). We may choose a particular solution $\rho(t) = u(t)$ of Milne's equation, and we may subsequently analyse the parametric oscillator solutions in terms of a fundamental pair of solutions

$$f_\pm(t) = u(t)\exp(\pm \mathrm{i}\alpha(t)), \qquad \dot{\alpha} = u^{-2}, \text{ with } W(f_+, f_-) = f_+\dot{f}_- - \dot{f}_+ f_- = -2\mathrm{i}. \tag{6}$$

Any parametric oscillator solution $x(t)$ clearly forms constant Wronskian relations with the two fundamental solutions (6). Consequently $x(t)$ is subject to two first-order constants of motion for the Ermakov system:

$$\ell_+ = W(x, f_+) = x\dot{f}_+ - \dot{x}f_+ = \left(x\dot{u} - \dot{x}u + \mathrm{i}\frac{x}{u}\right)\mathrm{e}^{\mathrm{i}\alpha}, \tag{7}$$

$$\ell_- = W(x, f_-) = x\dot{f}_- - \dot{x}f_- = \left(x\dot{u} - \dot{x}u - \mathrm{i}\frac{x}{u}\right)\mathrm{e}^{-\mathrm{i}\alpha}. \tag{8}$$

Note that similar invariants were discussed in relation to coupled parametric oscillators as well [12, 13]. From these first-order invariants the second-order Ermakov–Lewis invariant is obtained as

$$\mathcal{L} = \frac{1}{2}\ell_+\ell_- = \frac{1}{2}\left((x\dot{u} - \dot{x}u)^2 + \left(\frac{x}{u}\right)^2\right). \tag{9}$$

Whilst the invariants ℓ_+, ℓ_-, and $\mathcal{L}$ involve solutions of both equations of the Ermakov system, we now turn to invariants relating any two solutions $u(t)$ and $v(t)$ of the nonlinear Milne equation (2).

With a second pair of fundamental solutions for the parametric oscillator equation (1), given by

$$g_\pm(t) = v(t)\exp(\pm \mathrm{i}\beta(t)), \qquad \dot{\beta} = v^{-2}, \text{ with } W(g_+, g_-) = g_+\dot{g}_- - \dot{g}_+ g_- = -2\mathrm{i}, \tag{10}$$

we consider the Wronskians $W(f_\pm, g_\pm)$ between the pairs of fundamental solutions, which will imply new relations for the Milne solutions $u(t)$ and $v(t)$. Hence, we obtain the first-order Milne invariants

$$m_{++} = W(g_+, f_+) = \left(v\dot{u} - \dot{v}u + \mathrm{i}\left(\frac{v}{u} - \frac{u}{v}\right)\right)\mathrm{e}^{\mathrm{i}(\beta+\alpha)}, \tag{11}$$

$$m_{--} = W(g_-, f_-) = \left(v\dot{u} - \dot{v}u - \mathrm{i}\left(\frac{v}{u} - \frac{u}{v}\right)\right)\mathrm{e}^{-\mathrm{i}(\beta+\alpha)}, \tag{12}$$

$$m_{+-} = W(g_+, f_-) = \left(v\dot{u} - \dot{v}u - \mathrm{i}\left(\frac{v}{u} + \frac{u}{v}\right)\right)\mathrm{e}^{\mathrm{i}(\beta-\alpha)}, \tag{13}$$

$$m_{-+} = W(g_-, f_+) = \left(v\dot{u} - \dot{v}u + \mathrm{i}\left(\frac{v}{u} + \frac{u}{v}\right)\right)\mathrm{e}^{-\mathrm{i}(\beta-\alpha)}. \tag{14}$$

They appear in obvious conjugate pairs. The exponentials containing the phases in the above Milne invariants can thus be eliminated for each conjugate pair by forming two second-order invariants. On multiplying the first pair we form

$$\mathcal{M}^{(1)} = \frac{1}{2}m_{++}m_{--} = \frac{1}{2}\left((v\dot{u} - \dot{v}u)^2 + \left(\frac{v}{u} - \frac{u}{v}\right)^2\right), \tag{15}$$

and by multiplying the last pair we obtain

$$\mathcal{M}^{(2)} = \frac{1}{2}m_{+-}m_{-+} = \frac{1}{2}\left((v\dot{u} - \dot{v}u)^2 + \left(\frac{v}{u} + \frac{u}{v}\right)^2\right). \tag{16}$$

The two second-order invariants obtained here have a similar form as has the Ermakov–Lewis invariant and we see the trivial relation

$$\mathcal{M}^{(2)} = \mathcal{M}^{(1)} + 2. \tag{17}$$

Other second-order Milne invariants, which contain one of the two phases at a time, are also possible to construct. Combining (12) and (13), with a normalization as in (7) and (8), we get

$$\mathcal{A}_M^{(-)} = m_{--}m_{+-} = \left((v\dot{u} - \dot{v}u)^2 - \left(\frac{v}{u}\right)^2 + \left(\frac{u}{v}\right)^2 - 2\mathrm{i}(v\dot{u} - \dot{v}u)\frac{v}{u}\right)\mathrm{e}^{-2\mathrm{i}\alpha}, \tag{18}$$

which indicate a relation between the single accumulated phase $\alpha(t)$ and the local behaviours of $u(t)$ and $v(t)$. Furthermore, a conjugate invariant is derived from (11) and (14):

$$\mathcal{A}_M^{(+)} = m_{-+}m_{++} = \left((v\dot{u} - \dot{v}u)^2 - \left(\frac{v}{u}\right)^2 + \left(\frac{u}{v}\right)^2 + 2\mathrm{i}(v\dot{u} - \dot{v}u)\frac{v}{u}\right)\mathrm{e}^{2\mathrm{i}\alpha}. \tag{19}$$

In the same way we obtain a second pair of invariants involving the single phase $\beta(t)$:

$$\mathcal{B}_M^{(-)} = m_{--}m_{-+} = \left((v\dot{u} - \dot{v}u)^2 + \left(\frac{v}{u}\right)^2 - \left(\frac{u}{v}\right)^2 + 2\mathrm{i}(v\dot{u} - \dot{v}u)\frac{u}{v}\right)\mathrm{e}^{-2\mathrm{i}\beta}, \tag{20}$$

$$\mathcal{B}_M^{(+)} = m_{+-}m_{++} = \left((v\dot{u} - \dot{v}u)^2 + \left(\frac{v}{u}\right)^2 - \left(\frac{u}{v}\right)^2 - 2\mathrm{i}(v\dot{u} - \dot{v}u)\frac{u}{v}\right)\mathrm{e}^{2\mathrm{i}\beta}. \tag{21}$$

Again, in analogy with the construction of the Ermakov–Lewis invariant (9), it is possible to eliminate the phases and, as a result here, obtain fourth-order invariants. It turns out that both pairs, $\mathcal{A}_M^{(\pm)}$ and $\mathcal{B}_M^{(\pm)}$, combine to the same fourth-order invariant:

$$\begin{aligned}\mathcal{T}_M &= \tfrac{1}{2}\mathcal{A}_M^{(+)}\mathcal{A}_M^{(-)} = \tfrac{1}{2}\mathcal{B}_M^{(+)}\mathcal{B}_M^{(-)} \\ &= \frac{1}{2}\left(\left((v\dot{u} - \dot{v}u)^2 + \left(\frac{v}{u}\right)^2 - \left(\frac{u}{v}\right)^2\right)^2 + \left(2(v\dot{u} - \dot{v}u)\frac{u}{v}\right)^2\right).\end{aligned} \tag{22}$$

The sets of invariants $\{\mathcal{A}_M^{(+)}, \mathcal{A}_M^{(-)}, \mathcal{T}_M\}$ and $\{\mathcal{B}_M^{(+)}, \mathcal{B}_M^{(-)}, \mathcal{T}_M\}$ show an interesting similarity with the Ermakov–Lewis invariants $\{\ell_+, \ell_-, \mathcal{L}\}$, derived in equations (8) and (9). It is clear that the invariant $\mathcal{T}_M$ can also be obtained from $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$, or directly from the basic first-order m invariants in equations (11)–(14).

In this note it is shown that various classical invariants relevant for uncoupled Ermakov systems follow from Wronskian constants of the parametric oscillator system alone. The invariants pertaining to the Milne oscillator alone seem to have been mainly unexplored (see, however, [14, 15]).

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